Exercise 2.2.13

(Terminal velocity) The velocity $v(t)$ of a skydiver falling to the ground is governed by

$$m\dot{v} = mg - kv^2$$

where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.

b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the *terminal velocity*. (Beware of bad jokes about the word *terminal* and parachutes that fail to open.)

c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

d) An experimental study (Carlson et al. 1942) confirmed that that the equation $m\dot{v} = mg - kv^2$ gives a good quantitative fit to data on human skydivers. Six men were dropped from altitudes varying from 10,600 feet to 31,400 feet to a terminal altitude of 2,100 feet, at which they opened their parachutes. The long free fall from 31,400 to 2,100 feet took 116 seconds. The average weight of the men and their equipment was 261.2 pounds. In these units, $g = 32.2$ ft/sec^2. Compute the average velocity V_{avg} for the long free fall.

e) Using the data given here, estimate the terminal velocity, and the value of the drag constant k. (Hints: First you need to find an exact formula for $s(t)$, the distance fallen, where $s(0) = 0$, $\dot{s} = v$, and $v(t)$ is known from part (a). You should get $s(t) = \frac{v^2}{g}\ln\left(\cosh\frac{gt}{V}\right)$, where V is the terminal velocity. Then solve for V graphically or numerically, using $s = 29{,}300$, $t = 116$, and $g = 32.2$.)
A slicker way to estimate V is to suppose $V \approx V_{\text{avg}}$ for the long free fall, as a rough first approximation. Then show that $gt/V \approx 15$. Since $gt/V \gg 1$, we may use the approximation $\ln(\cosh x) \approx x - \ln 2$ for $x \gg 1$. Derive this approximation and then use it to obtain an analytical estimate of V. Then k follows from part (b).

This analysis is from Davis (1962).

Solution

Part a)

Here the aim is to solve the following initial value problem.

$$m\frac{dv}{dt} = mg - kv^2, \quad v(0) = 0$$

Solve the ODE by separating variables and integrating both sides.

$$\frac{m}{mg - kv^2}\,dv = dt$$

$$\frac{1}{g - \frac{k}{m}v^2}\,dv = dt$$

$$\int^v \frac{du}{g - \frac{k}{m}u^2} = \int dt \tag{1}$$

Make a trigonometric substitution to integrate the left side.

$$u = \sqrt{\frac{mg}{k}}\sin\theta \quad \rightarrow \quad g - \frac{k}{m}u^2 = g - \frac{k}{m}\frac{mg}{k}\sin^2\theta = g(1-\sin^2\theta) = g\cos^2\theta$$
$$du = \sqrt{\frac{mg}{k}}\cos\theta\, d\theta$$

As a result, equation (1) becomes

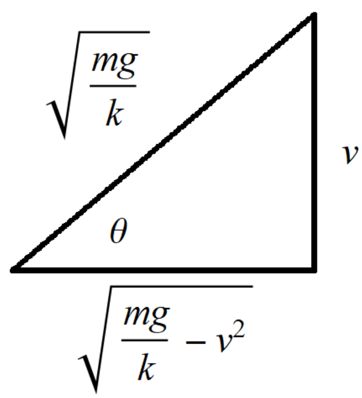
$$\int^{\sin^{-1}\left(\sqrt{\frac{k}{mg}}v\right)} \frac{\sqrt{\frac{mg}{k}}\cos\theta\, d\theta}{g\cos^2\theta} = t + C$$

$$\sqrt{\frac{m}{gk}}\int^{\sin^{-1}\left(\sqrt{\frac{k}{mg}}v\right)} \sec\theta\, d\theta = t + C$$

$$\sqrt{\frac{m}{gk}}\ln|\sec\theta + \tan\theta|\Big|^{\sin^{-1}\left(\sqrt{\frac{k}{mg}}v\right)} = t + C$$

$$\sqrt{\frac{m}{gk}}\ln\left|\sec\sin^{-1}\left(\sqrt{\frac{k}{mg}}v\right) + \tan\sin^{-1}\left(\sqrt{\frac{k}{mg}}v\right)\right| = t + C. \tag{2}$$

Draw the implied right triangle.



Use it to determine $\sec\theta$ and $\tan\theta$.

$$\sec\theta = \frac{\sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k} - v^2}} \quad \text{and} \quad \tan\theta = \frac{v}{\sqrt{\frac{mg}{k} - v^2}}$$

Consequently, equation (2) becomes

$$\sqrt{\frac{m}{gk}}\ln\left|\frac{\sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k}-v^2}}+\frac{v}{\sqrt{\frac{mg}{k}-v^2}}\right|=t+C$$

$$\sqrt{\frac{m}{gk}}\ln\left|\frac{\sqrt{\frac{mg}{k}}+v}{\sqrt{\frac{mg}{k}-v^2}}\right|=t+C. \tag{3}$$

Now apply the initial condition to determine C.

$$\sqrt{\frac{m}{gk}}\ln\left|\frac{\sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k}}}\right|=C \quad \rightarrow \quad C=0$$

Equation (3) then becomes

$$\sqrt{\frac{m}{gk}}\ln\left|\frac{\sqrt{\frac{mg}{k}}+v}{\sqrt{\frac{mg}{k}-v^2}}\right|=t$$

$$\ln\left|\frac{\sqrt{\frac{mg}{k}}+v}{\sqrt{\frac{mg}{k}-v^2}}\right|=\sqrt{\frac{gk}{m}}t$$

$$\left|\frac{\sqrt{\frac{mg}{k}}+v}{\sqrt{\frac{mg}{k}-v^2}}\right|=\exp\left(\sqrt{\frac{gk}{m}}t\right).$$

Place $\pm$ on the right side to remove the absolute value sign.

$$\frac{\sqrt{\frac{mg}{k}}+v}{\sqrt{\frac{mg}{k}-v^2}}=\pm\exp\left(\sqrt{\frac{gk}{m}}t\right)$$

Choose the plus sign so that when $v=0$ and $t=0$ are plugged in, the equation is a true statement.

$$\frac{\sqrt{\frac{mg}{k}}+v}{\sqrt{\frac{mg}{k}-v^2}}=\exp\left(\sqrt{\frac{gk}{m}}t\right)$$

Square both sides.

$$\frac{\frac{mg}{k}+2\sqrt{\frac{mg}{k}}v+v^2}{\frac{mg}{k}-v^2}=\exp\left(2\sqrt{\frac{gk}{m}}t\right)$$

Solve for v.

$$\frac{mg}{k} + 2\sqrt{\frac{mg}{k}}v + v^2 = \frac{mg}{k}\exp\left(2\sqrt{\frac{gk}{m}}t\right) - v^2\exp\left(2\sqrt{\frac{gk}{m}}t\right)$$

$$\left[1 + \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right]v^2 + 2\sqrt{\frac{mg}{k}}v + \frac{mg}{k}\left[1 - \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right] = 0$$

$$v(t) = \frac{-2\sqrt{\frac{mg}{k}} \pm \sqrt{\frac{4mg}{k} - \frac{4mg}{k}\left[1 - \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right]\left[1 + \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right]}}{2\left[1 + \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right]}$$

$$v(t) = \frac{-2\sqrt{\frac{mg}{k}} \pm \sqrt{\frac{4mg}{k}\exp\left(4\sqrt{\frac{gk}{m}}t\right)}}{2\left[1 + \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right]}$$

In order to have $v(0) = 0$, choose the plus sign.

$$v(t) = \frac{-2\sqrt{\frac{mg}{k}} + \sqrt{\frac{4mg}{k}\exp\left(4\sqrt{\frac{gk}{m}}t\right)}}{2\left[1 + \exp\left(2\sqrt{\frac{gk}{m}}t\right)\right]}$$

$$v(t) = 2\sqrt{\frac{mg}{k}}\frac{\exp\left(2\sqrt{\frac{gk}{m}}t\right) - 1}{2\left[\exp\left(2\sqrt{\frac{gk}{m}}t\right) + 1\right]}$$

$$v(t) = \sqrt{\frac{mg}{k}}\left[\frac{1 - \exp\left(-2\sqrt{\frac{gk}{m}}t\right)}{1 + \exp\left(-2\sqrt{\frac{gk}{m}}t\right)}\right]$$

Therefore,

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right).$$

Part b)

The terminal velocity is

$$\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}}.$$

Part c)

Here a graphical analysis will be made for the initial value problem.

$$\dot{v} = g - \frac{k}{m}v^2$$

The fixed points occur where $\dot{v} = 0$.

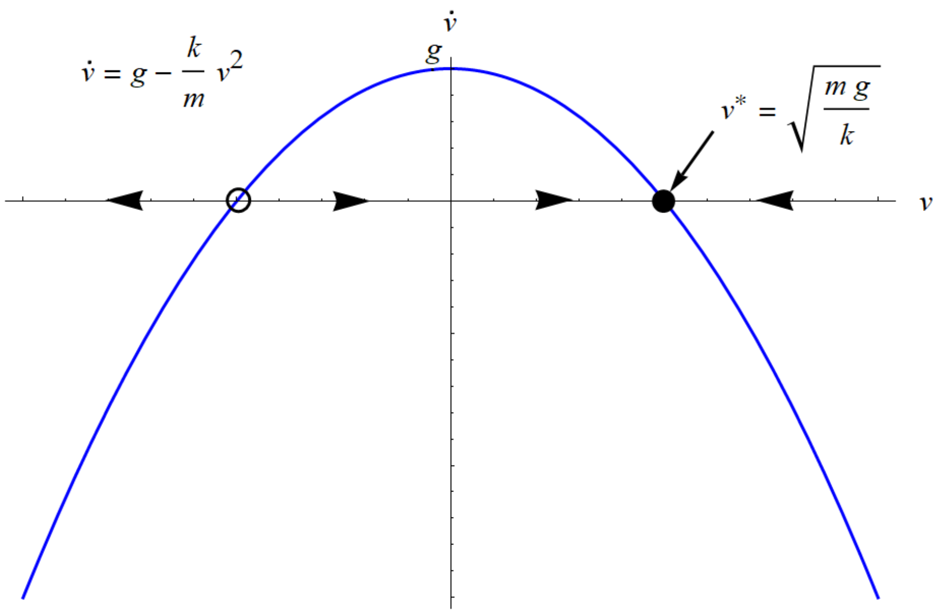
$$g - \frac{k}{m}v^{*2} = 0$$

$$v^* = \pm\sqrt{\frac{mg}{k}}$$

The plus sign is chosen since the skydiver falls in the direction of increasing position.

$$v^* = \sqrt{\frac{mg}{k}}$$

Draw the graph of $\dot{v}$ versus v to determine whether the fixed point is stable or unstable.



When the function is negative the flow is to the left, and when the function is positive the flow is to the right. This makes the fixed point at $v^* = \sqrt{mg/k}$ locally stable.

Part d)

The average velocity is the distance travelled over the time it takes.

$$V_{\text{avg}} = \frac{\Delta x}{\Delta t} = \frac{31\,400 - 2100}{116}\ \frac{\text{ft}}{\text{s}} \approx 253\ \frac{\text{ft}}{\text{s}}$$

Part e)

Begin with the result of part (a).

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} t\right)$$

Integrate both sides with respect to t from 0 to 116.

$$\int_0^{116} v(t)\,dt = \int_0^{116} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} t\right) dt$$

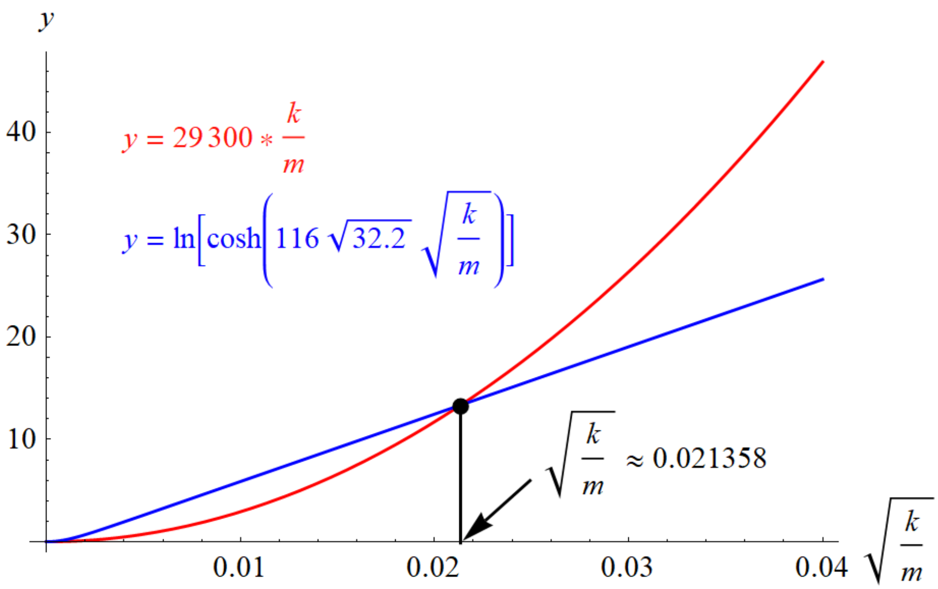
$$s(116) - s(0) = \sqrt{\frac{mg}{k}} \int_0^{116} \tanh\left(\sqrt{\frac{gk}{m}} t\right) dt$$

$$(31\,400 - 2100) - 0 = \sqrt{\frac{mg}{k}} \cdot \sqrt{\frac{m}{gk}} \ln\left[\cosh\left(\sqrt{\frac{gk}{m}} t\right)\right]\Bigg|_0^{116}$$

$$29\,300 = \frac{m}{k} \ln\left[\cosh\left(116\sqrt{\frac{gk}{m}}\right)\right]$$

$$29\,300\frac{k}{m} = \ln\left[\cosh\left(116\sqrt{g}\sqrt{\frac{k}{m}}\right)\right] \tag{4}$$

Plot the functions on both sides versus $\sqrt{k/m}$ and see where the curves intersect.



Based on this graph, the numerical solution to equation (4) is

$$\sqrt{\frac{k}{m}} \approx 0.021358.$$

Solve for k.

$$k \approx 0.021358^2 m$$

Therefore, since $m \approx 261.2/32.2$ slugs ≈ 8.11 slugs,

$$k \approx 0.021358^2 \left(\frac{261.2}{32.2}\right) \approx 3.70 \times 10^{-3}\ \frac{\text{lb} \cdot \text{s}^2}{\text{ft}^2},$$

and the terminal velocity is

$$\sqrt{\frac{mg}{k}} \approx \sqrt{\frac{261.2}{3.70 \times 10^{-3}}}\ \frac{\text{ft}}{\text{s}} \approx 266\ \frac{\text{ft}}{\text{s}}.$$